AN OVERVIEW OF THE PANJER METHOD FOR DERIVING THE AGGREGATE CLAIMS DISTRIBUTION

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Harry H. Panjer derives a recursive method for deriving the aggregate distribution of claims in his article "The Aggregate Claims Distribution and Stop-Loss Reinsurance" published in the *Transactions of the Society of Actuaries*, Volume XXXII, 1980, pages 523 – 545. The Panjer method may be viewed as an alternative to more traditional Monte Carlo simulation methods as well as an alternative to risk theory methods involving convolutions of the distribution of claim amounts given a certain number of claims. The chief advantage of the Panjer method is that the complete aggregate distribution of claims for a given block of policies may be quickly and directly calculated, rather than estimated using an increasing number of Monte Carlo trials or directly calculated using a time-consuming set of numerical convolutions. By extension, the Panjer method is also useful tool when setting a company's retention level and in developing stop-loss reinsurance programs.

Panjer develops the necessary theoretical framework to recursively calculate the complete aggregate distribution of claims in the context of a block of group life certificates exposed to the force of mortality over a single time period. This framework may easily be extended to the context of a block of individually underwritten policies. Regardless of group or individual contexts, a discrete distribution of amounts of insurance and the sum of the forces of mortality over each grouping of policies or certificates are the only inputs necessary to develop the aggregate distribution of claims.

Basic assumptions made within the framework of the Panjer article include:

- ✓ Each life is insured for *a fixed amount of death benefit* (although Panjer extends his theory to a variable amount of death benefit within the same article).

- ✓ Each policy is *independent*, and *those lives dying* during the time period examined (typically a one-year) *are replaced immediately* by lives of identical risk. Under the collective risk model, the number of claims arising from is a Poisson-distributed random variable.

- ✓ To minimize the number of computations, a convenient unit of insurance should be selected and individual policies grouped into *unique issue amount categories* that are integral multiples of the chosen unit of insurance.

- ✓ The sum of the forces of mortality for all lives, by category in force. This may be approximated for each category separately as the expected annual mortality times the number of policies in force.

To better illustrate the Panjer method, consider a group of 1,000 policies summarized in Table 1 that were selected from a hypothetical block of individually underwritten term life insurance policies. Each insured was age 45 at issue and is now attained age 49.

TABLE 1
Hypothetical Distribution of Term Life Policies, Issue Age 45

Amount of Insurance	Number of Policies	Average Size	Insurance In Force	Expected Mortality
less than $10,000	5	$ 6,000	$ 30,000	0.002188
$10,000	62	$ 10,000	$ 620,000	0.002188
$10,001 - $24,999	13	$ 24,000	$ 312,000	0.002188
$25,000	105	$ 25,000	$ 2,625,000	0.001838
$25,001 - $49,999	22	$ 47,000	$ 1,034,000	0.001838
$50,000	156	$ 50,000	$ 7,800,000	0.001794
$50,001 - $74,999	30	$ 74,000	$ 2,220,000	0.001794
$75,000	223	$ 75,000	$ 16,725,000	0.001750
$75,001 - $99,999	89	$ 99,000	$ 8,811,000	0.001750
$100,000	295	$ 100,000	$ 29,500,000	0.001663
Total	1000	$ 69,677	$ 69,677,000	0.001730

The task of grouping policies into issue amount categories may seem daunting, especially for large blocks of business. However, there are two mitigating factors that may make the first time through this process less onerous:

- For many products, consumer purchases tend to be clustered around discrete issue amounts. As illustrated in Table 1, the cluster issue amounts of $25,000, $50,000, $75,000, and $100,000 represent 84.1% of the policies in force.

- Those policies that fall outside the clusters of issue amount, when grouped into issue amount bands, often have average sizes near the top of their band. As illustrated in Table 1, the average size of the policies falling in the issue amount band $75,001 to $99,999 is $99,000. Often, this may result from the desire of certain insureds to apply for the maximum amount of insurance available for a minimum amount of underwriting scrutiny.

Mathematically, Panjer defines the following terms:

U = the unit of insurance

i = one unique issue amount category, measured in units of insurance

n = the largest unique issue amount category, measured in units of insurance

iU	=	the amount of insurance in issue amount category i, measured in dollars of insurance
nU	=	the amount of insurance in the largest issue amount category, measured in dollars of insurance
Y_i	=	the random variable representing the number of claims of amount iU
θ_i	=	the sum of the forces of mortality for all lives in issue amount category i.
	≈	(Number of Policies in category i) * (Expected Mortality for category i), where Expected Mortality is a proxy for the force of mortality
X_i	=	the random variable representing the aggregate claims of amount iU
X	=	$X_1 + X_2 + \ldots + X_n$, the aggregate claims over all issue amount categories
P_i	=	the probability that the aggregate claims will be exactly iU
	=	$\Pr\{X = iU\}$

Since the sum of independent Poisson random variables is itself a Poisson random variable, Y_i is Poisson-distributed with parameter θ_i. Dropping the issue amount categories i where $\theta_i = 0$, Panjer uses the probability generating function of X to develop a recursive relation

$$P_i = (1/i) \sum_{\substack{j=1 \\ \theta_i \neq 0}}^{\min(i,n)} j\theta_j \, P_{i-j}, \qquad (1)$$

where, in the case when i = 0 (the probability of $0 claims),

$$P_0 = \Pr\{X = 0\} = \exp\Bigl(-\sum_{\substack{j=1 \\ \theta_i \neq 0}}^{n} \theta_j\Bigr). \qquad (2)$$

For the policies in Table 1, the largest common divisor of the average sizes, $1,000, would be a natural choice for the unit of insurance U. This would result in 10 issue amount categories i, including $6, $10, $24, $25, $47, $50, $74, $75, $99, and $100. The largest issue amount category, n, would be $100. Table 2 reformats the information provided in Table 1 for use with the Panjer method.

TABLE 2

Inputs to the Panjer Method

Category	Number of Policies	Expected Mortality	Theta
6	5	0.002188	0.0109375
10	62	0.002188	0.1356250
24	13	0.002188	0.0284375
25	105	0.001838	0.1929375
47	22	0.001838	0.0404250
50	156	0.001794	0.2798250
74	30	0.001794	0.0538125
75	223	0.001750	0.3902500
99	89	0.001750	0.1557500
100	295	0.001663	0.4904375
Total	1000		1.7784375

As a practical exercise, let's compute the aggregate claim distribution using the Panjer method and the information provided in Table 2. First compute P_0, the probability of $0 claims, which is given by formula (2)

$$P_0 = \exp\left(- \sum_{\substack{j=1 \\ \theta_i \neq 0}}^{n} \theta_j \right) = \exp(-1.7784375) = 0.1689019$$

Then, using formula (1) recursively,

$P_6 = (1/6)(6 * \theta_6 * P_0) = (1/6)(6 * 0.0109375)(0.1689019) = 0.00184736$

$P_{10} = (1/10)(10 * \theta_{10} * P_0) = (1/10)(10 * 0.1356250)(0.1689019) = 0.02290732$

$P_{12} = (1/12)(6 * \theta_6 * P_6) = (1/12)(6 * 0.0109375)(0.00184736) = 0.00001010$

and so on to produce the complete aggregate distribution of claims by amount of insurance.

By slightly changing the inputs to the Panjer method, a complete aggregate distribution of claims by number can be produced. This was accomplished by resetting the categories to values at or very near the average size of the entire group of Age 45 policies. Since the values of each category must be unique, integral values of the chosen unit, care must be taken during this process. The average size policy for all policies is $69,677 (from Table 1). Table 3 presents the restated inputs to the Panjer method necessary for a by number analysis of mortality, with U = $100.

TABLE 3
Inputs to the Panjer Method (By Number Study)

Category	Number of Policies	Expected Mortality	Theta
690	5	0.002188	0.0109375
691	62	0.002188	0.1356250
692	13	0.002188	0.0284375
693	105	0.001838	0.1929375
694	22	0.001838	0.0404250
695	156	0.001794	0.2798250
696	30	0.001794	0.0538125
697	223	0.001750	0.3902500
698	89	0.001750	0.1557500
699	295	0.001663	0.4904375
Total	1000		1.7784375

Table 4 summarizes the percentile distribution of actual-to-expected mortality ratios, by amount of insurance and by number of policies in force, as calculated using the Panjer method (programmed using Microsoft Visual Basic and Microsoft Excel) for the policies in Tables 2 and 3. Table 5 summarizes the complete probability density function and cumulative distribution function, by amount and by number, produced by the Panjer method.

TABLE 4
Percentile Distribution of Actual-to-Expected Mortality – Panjer Method

Selected Percentiles	Actual / Expected Mortality (by Amount)	Actual / Expected Mortality (by Number)
1.0%	0.00%	0.00%
5.0%	0.00%	0.00%
10.0%	0.00%	0.00%
15.0%	0.00%	0.00%
20.0%	19.86%	55.89%
25.0%	40.71%	56.07%
30.0%	49.10%	56.22%
35.0%	61.74%	56.28%
40.0%	70.00%	56.39%
45.0%	82.22%	56.44%
50.0%	82.62%	112.05%
55.0%	90.49%	112.25%
60.0%	103.34%	112.42%
65.0%	123.74%	112.58%
70.0%	143.57%	112.74%
75.0%	144.99%	168.13%
80.0%	165.23%	168.57%
85.0%	185.67%	168.87%
90.0%	207.29%	224.25%
91.0%	226.39%	224.55%
92.0%	227.38%	224.74%
93.0%	228.05%	224.90%
94.0%	247.40%	225.06%
95.0%	248.54%	225.23%
96.0%	268.30%	225.46%
97.0%	288.42%	280.70%
98.0%	309.44%	281.24%
99.0%	331.74%	282.08%
99.9%	443.14%	393.95%

Key Statistics	By Amount	By Number
mean claim amount	$ 120,533	$ 123,817
mean number of claims	1.73	1.78
std. dev. of claim amount	$ 99,300	$ 92,846
std. dev. of number of claims	1.43	1.33

TABLE 5
Aggregate Claim Distribution – Panjer Method

	Amount	Number
mean:	$ 120,533	1.73
std deviation:	$ 99,300	1.43

	Amount	Number
mean:	$ 123,817	1.78
std deviation:	$ 92,846	1.33

DISTRIBUTION BY CLAIM AMOUNT			
Claim Amount		Probability	Cumulative Probability
$	-	0.16890185	0.16890185
$	6,000	0.00184736	0.17074921
$	10,000	0.02290731	0.19365653
$	12,000	0.00001010	0.19366663
$	16,000	0.00025055	0.19391718
$	18,000	0.00000004	0.19391722
$	20,000	0.00155340	0.19547062
$	22,000	0.00000137	0.19547199
$	24,000	0.00480315	0.20027513
$	25,000	0.03258750	0.23286264
$	26,000	0.00001699	0.23287963
$	28,000	0.00000000	0.23287963
$	30,000	0.00012276	0.23300239
$	31,000	0.00035643	0.23335882
$	32,000	0.00000009	0.23335891
$	34,000	0.00065143	0.23401034
$	35,000	0.00441968	0.23843002
$	36,000	0.00000106	0.23843107
$	37,000	0.00000195	0.23843302
	:	:	:
	:	:	:
$	1,065,000	7.53449E-12	0.99999999
$	1,066,000	1.40235E-11	0.99999999
$	1,067,000	2.84348E-11	0.99999999
$	1,068,000	5.6554E-11	0.99999999
$	1,069,000	1.07301E-10	0.99999999
$	1,070,000	1.96111E-10	0.99999999
$	1,071,000	3.41652E-10	0.99999999
$	1,072,000	5.27873E-10	1.00000000

DISTRIBUTION BY NUMBER OF CLAIMS		
Number of Claims	Probability	Cumulative Probability
0	0.16890185	0.16890185
1	0.30038138	0.46928323
2	0.26710476	0.73638799
3	0.15834304	0.89473103
4	0.07040080	0.96513183
5	0.02504068	0.99017252
6	0.00742222	0.99759473
7	0.00188571	0.99948044
8	0.00041920	0.99989964
9	0.00008284	0.99998248
10	0.00001473	0.99999721
11	0.00000238	0.99999959
12	0.00000035	0.99999994
13	0.00000005	0.99999999
14	0.00000001	1.00000000